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Dynamic behaviors of Monod type chemostat model with impulsive perturbation on the nutrient concentration

Shulin Sun* and Lansun Chen

Department of Applied Mathematics, Dalian University of Technology, Liaoning, Dalian 116024, P.R.China
E-mail: sunsl_2004@yahoo.com.cn

Shulin Sun

School of Mathematics and Computer Science, Shanxi Normal University, Shanxi, Linfen 041004, P.R. China

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In this paper, the dynamic behaviors of a Monod type chemostat model with impulsive perturbation are investigated. Using Floquet theory and small amplitude perturbation method, we prove that the microorganism-eradication periodic solution is asymptotically stable if the impulsive period satisfies some conditions. Moreover, the permanence of the system is discussed in detail. Finally, we verify the main results by numerical simulation.

KEY WORDS: chemostat, impulsive input, extinction, permanence

AMS subject classification: 34K45, 34K60, 92D25, 92D40

1. Introduction

The chemostat is a simple and well-adopted laboratory apparatus used to culture microorganisms. It can be used to investigate microbial growth and has the advantage that the parameters are easily measurable. Sterile growth medium enters the chemostat at a constant rate; the volume within the chemostat is preserved by allowing excess medium to flow out through a siphon. We inoculate this chemostat with a heterotrophic bacterium that finds, in the medium, a lot of all necessary nutrients but one. This last nutrient is the limiting substrate. In [1], Smith and Waltman describe a chemostat and formulate various mathematical chemostat models. The specific growth rate of bacteria saturates at sufficiently high-substrate concentration. The functional response of the bacterium on the

*Corresponding author.

substrate is commonly assumed to be of the Monod type. Equations of the basic model take the form [2]

$$
\begin{aligned}
S^{'}(t) &= Q(S^0 - S(t)) - \frac{\mu_m S(t)x(t)}{\delta(K_m + S(t))},\\
x^{'}(t) &= x(t)(\frac{\mu_m S(t)}{K_m + S(t)} - Q),
\end{aligned} \tag{1.1}
$$

where the state variables $S(t)$ and $x(t)$ denote the concentrations of the limiting substrate and the microorganism at time t, respectively; S^0 denotes the input concentration of the limiting substrate per unit of time; δ denotes the yield of the microorganism per unit mass of substrate; Q is the dilution rate of the chemostat. The function $p(S) \triangleq \frac{\mu_m S(t)}{K_m + S(t)}$ denotes the microbial growth rate.

The dynamic behaviors of the basic model (1.1) are simple. The microorganism can either become extinct or persist at the positive equilibrium. The results depend on two parameters m and λ, where $m = \mu_m/Q$ and $\lambda = \frac{K_m}{S^0(m-1)}$. If $m \leqslant 1$ or $m > 1$ and $\lambda \geqslant 1$, then the microorganism becomes extinct. If $m > 1$ and $\lambda < 1$, then the microorganism persists.

In recent years, the microbial continuous culture has been investigated in [3–6] and some interesting results were obtained. Many scholars pointed out that it was necessary and important to consider models with periodic perturbations, since these models might be quite naturally exposed in many real world phenomena (for instance, food supply, mating habits, harvesting). In fact, almost perturbations occur in a more-or-less periodic fashion. However, there are some other perturbations such as fires, floods, and drainage of sewage which are not suitable to be considered continually. These perturbations bring sudden changes to the system. Systems with sudden perturbations are involving in impulsive differential equations, which have been studied intensively and systematically in [7,8]. Authors, in [9–12], introduced some impulsive differential equations in population dynamics and obtained some interest results. The research on the chemostat model with impulsive perturbations is not too much yet (see Refs. [13,14] and references therein). However, this is an interest problem in mathematical biology and laboratory experiment.

In this paper, we investigate how the impulsive perturbation of the substrate affects the dynamic behaviors of the chemostat continuous system. The chemostat model with impulsive perturbation is written as:

$$
\begin{aligned}
&S^{'}(t) = -QS(t) - \frac{\mu_m S(t)x(t)}{\delta(K_m + S(t))}, && t \neq nT,\\
&x^{'}(t) = x(t)\left(\frac{\mu_m S(t)}{K_m + S(t)} - Q\right), && t \neq nT,\\
&S(nT^+) = S(nT) + \tau S^0, \quad x(nT^+) = x(nT), && n = 1, 2, \ldots,\\
&S(0^+) \geqslant 0, \ \ x(0^+) \geqslant 0,
\end{aligned} \tag{1.2}
$$

where the first and second equations hold between pulses, the third equation describes the actual pulsing. $S(t), x(t)$, and other parameters are the same as (1.1). $T = \tau/Q$ is the period of the pulsing, τS^0 is the amount of limiting substrate pulsed each T. QS^0 units of substrate are added, on average, per unit of time.

2. Preliminary results

Let $R_+ = [0, +\infty)$, $R_+^2 = \{z \in R^2 : z \geqslant 0, z = (S, x)\}$, $\Omega = \text{int}R_+^2$, N be the set of nonnegative integers. Denote $f = (f_1, f_2)^T$ the map defined by the right hand of the anterior two equations of system (1.2).

Let V: $R_+ \times R_+^2 \to R_+$. Then V is said to belong to class V_0 if

(i) V is continuous in $(nT, (n+1)T] \times R_+^2$ and for each $z \in R_+^2$, $n \in N$, $\lim_{(t,y)\to(nT^+,z)} V(t, y) = V(nT^+, z)$ exists;

(ii) V is locally Lipschitzian in z.

Definition 2.1. Let $V \in V_0$, $(t, z) \in (nT, (n+1)T] \times R_+^2$. The upper right derivative of $V(t, z)$ with respect to the impulsive differential system (1.2) is defined as

$$D^+V(t, z) = \lim \sup_{h\to 0^+} \frac{1}{h}[V(t+h, z+hf(t, z)) - V(t, z)].$$

The solution of system (1.2) is a piecewise continuous function $z(t) = (S(t), x(t))$: $R_+ \to R_+^2$, $z(t)$ is continuous on $(nT, (n+1)T], n \in N$ and $z(nT^+) = \lim_{t\to nT^+} z(t)$ exists. Obviously, the global existence and uniqueness of solutions of the system (1.2) is guaranteed by the smoothness properties of f (see Ref. [7] for details). Hence, we have the following lemma.

Lemma 2.1. Suppose $z(t)$ is a solution of (1.2) with $z(0^+) \geqslant 0$. Then $z(t) \geqslant 0$ for all $t \geqslant 0$. Moreover, if $z(0^+) > 0$, then $z(t) > 0$ for all $t \geqslant 0$.

Definition 2.2. System (1.2) is said to be permanent if there exist constants $M \geqslant m > 0$ such that $m \leqslant S(t) \leqslant M, m \leqslant x(t) \leqslant M$ for t large enough, where $(S(t), x(t))$ is any solution of (1.2) with $S(0^+) > 0, x(0^+) > 0$.

Lemma 2.2. (Comparison Theory, [7, Theorem 3.1.1]) Let V: $R_+ \times R_+^2 \to R_+$ and $V \in V_0$. Assume that

$$\begin{aligned} &D^+V(t, z(t)) \leqslant g(t, V(t, z(t))), && t \neq nT, \\ &V(t, z(t^+)) \leqslant \psi_n(V(t, z(t))), && t = nT, \end{aligned} \tag{2.1}$$

where g: $R_+ \times R_+ \to R$ is continuous in $(nT, (n+1)T] \times R_+$ and for each $x \in R_+$, $n \in N$, $\lim_{(t,y)\to(nT^+,x)} g(t,y) = g(nT^+, x)$ exist; ψ_n: $R_+ \to R_+$ is nondecreasing. Let $r(t) = r(t, 0, u_0)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{array}{ll} u' = g(t,u), & t \neq nT, \\ u(t^+) = \psi_n(u(t)), & t = nT, \\ u(0^+) = u_0, & \end{array} \tag{2.2}$$

existing on $[0, \infty)$. Then $V(0^+, z_0) \leqslant u_0$ implies that $V(t, z(t)) \leqslant r(t), t \geqslant 0$, where $z(t) = z(t, 0, z_0)$ is any solution of (1.2) existing on $[0, \infty)$.

Remark. In Lemma 2.2, if the directions of the inequalities in (2.1) are reversed, that is,

$$\begin{array}{l} D^+V(t, z(t)) \geqslant g(t, V(t, z(t))),\ t \neq nT, \\ V(t, z(t^+)) \geqslant \psi_n(V(t, z(t))), \quad t = nT \end{array}$$

then $V(t, z(t)) \geqslant \rho(t), t \geqslant t_0$, where $\rho(t)$ is the minimal solution of (2.2) on $[0, \infty)$.

The function we will use is in the form $-Q\omega(t)$. For convenience, we give some basic properties of the following system

$$\begin{array}{ll} \omega'(t) = -Q\omega(t), & t \neq nT, \\ \triangle\omega(t) = \omega(t^+) - \omega(t) = \tau S^0, & t = nT, \\ \omega(0^+) = S(0^+) \geqslant 0. & \end{array} \tag{2.3}$$

Clearly,

$$\omega^*(t) = \frac{\tau S^0 e^{-Q(t-nT)}}{1 - e^{-QT}}, \qquad t \in \left(nT, (n+1)T], n \in N, \left(\omega^*(0^+) = \frac{\tau S^0}{1 - e^{-QT}}\right)\right)$$

is a positive periodic solution of (2.3). The solution of (2.3) is $\omega(t) = [\omega(0^+) - \omega^*(0^+)]e^{-Qt} + \omega^*(t)$, $t \in (nT, (n+1)T], n \in N$. Therefore, the following result holds.

Lemma 2.3. System (2.3) has a positive periodic solution $\omega^*(t)$ and $|\omega(t) - \omega^*(t)| \to 0$ as $t \to \infty$ for any solution $\omega(t)$ of (2.3). Moreover, $\omega(t) \geqslant \omega^*(t)$ if $\omega(0^+) \geqslant \omega^*(0^+)$ and $\omega(t) < \omega^*(t)$ if $\omega(0^+) < \omega^*(0^+)$.

3. Extinction and permanence

Obviously, system (1.2) has a T-periodic solution $(S^*(t), 0)$ at which microorganism culture fails, where

$$(S^*(t), 0) = \left(\frac{\tau S^0 e^{-Q(t-nT)}}{1 - e^{-QT}}, 0 \right), t \in (nT, (n+1)T].$$

Theorem 3.1. Periodic solution $(S^*(t), 0)$ is globally asymptotically stable if

$$\int_0^T \frac{\mu_m S^*(t)}{K_m + S^*(t)} dt < QT, \text{ i.e. } \frac{\mu_m}{Q} \ln \frac{K_m(1 - e^{-QT}) + \tau S^0}{K_m(1 - e^{-QT}) + \tau S^0 e^{-QT}} < QT.$$

Proof. The local asymptotic stability of periodic solution $(S^*(t), 0)$ may be determined by considering the behavior of small amplitude perturbation of the solution. Let $(S(t), x(t))$ be any solution of (1.2). We define $S(t) = u(t) + S^*(t), x(t) = v(t)$.

The corresponding linear system of (1.2) is

$$\begin{aligned}
&u'(t) = -Qu - \frac{1}{\delta} \frac{\mu_m S^*(t)}{K_m + S^*(t)} v, && t \neq nT, \\
&v'(t) = [\frac{\mu_m S^*(t)}{K_m + S^*(t)} - Q] v, && t \neq nT, \\
&u(t^+) = u(t), && t = nT, \\
&v(t^+) = v(t), && t = nT.
\end{aligned} \tag{3.1}$$

Let $\Phi(t)$ be a fundamental matrix of (3.1). Then $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} -Q & -\frac{1}{\delta} \frac{\mu_m S^*(t)}{K_m + S^*(t)} \\ 0 & \frac{\mu_m S^*(t)}{K_m + S^*(t)} - Q \end{pmatrix} \Phi(t) \triangleq A(t)\Phi(t) \tag{3.2}$$

and $\Phi(0) = I$, the identity matrix.

The resetting impulsive conditions of (3.1) becomes

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}.$$

The stability of the periodic solution $(S^*(t), 0)$ is determined by the eigenvalues of the monodromy matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Phi(T) = \Phi(T).$$

From (3.2), we can obtain $\Phi(T) = \Phi(0)\exp(\int_0^T A(t)\mathrm{d}t) \triangleq \Phi(0)\exp(\bar{A})$. Therefore, the Floquet multipliers of system (3.1) are

$$\mu_1 = \exp(-QT) < 1, \quad \mu_2 = \exp\left(\int_0^T \frac{\mu_m S^*(t)}{K_m + S^*(t)} dt - QT\right).$$

According to Floquet theory ([8, Theorem 3.5]), $(S^*(t), 0)$ is asymptotically stable if

$$|\mu_2| < 1, \text{ i.e. } \frac{\mu_m}{Q} \ln \frac{K_m(1 - \mathrm{e}^{-QT}) + \tau S^0}{K_m(1 - \mathrm{e}^{-QT}) + \tau S^0 e^{-QT}} < QT.$$

In the sequel, we prove the global attractability of periodic solution $(S^*(t), 0)$.

Note that $S'(t) \leqslant -QS(t)$ and the comparison system (2.3). We have $S(t) \leqslant \omega(t)$ and $S(t) \to \omega^*(t) = S^*(t)$ as $t \to \infty$ by lemmas 2.2 and 2.3.

From the condition of theorem 3.1, we can choose $\epsilon > 0$ small enough such that $\sigma = \int_0^T \frac{\mu_m(S^*(t) + \epsilon)}{K_m + S^*(t) + \epsilon}\mathrm{d}t - QT < 0$ and $S(t) \leqslant S^*(t) + \epsilon$ for t large enough. Without loss of generality, we can assume $S(t) \leqslant S^*(t) + \epsilon$ for all $t \geqslant 0$. From system (1.2), we have

$$x'(t) \leqslant x(t)\left[-Q + \frac{\mu_m(S^*(t) + \epsilon)}{K_m + S^*(t) + \epsilon}\right]. \tag{3.3}$$

Integrating (3.3) on $(nT, (n+1)T]$, we have

$$x((n+1)T) \leqslant x(nT^+)\exp\left(\int_{nT}^{(n+1)T} \left[-Q + \frac{\mu_m(S^*(t) + \epsilon)}{K_m + S^*(t) + \epsilon}\right]\mathrm{d}t\right) = x(nT)\exp(\sigma).$$

Therefore, $x(nT) \leqslant x(0^+)\exp(n\sigma)$ and $x(nT) \to 0$ as $n \to \infty$. Since $0 \leqslant x(t) \leqslant x(nT)\exp(\sigma) \leqslant x(0^+)\exp((n+1)\sigma)$ for any $t \in (nT, (n+1)T]$, we have $x(t) \to 0$ as $t \to \infty$. The proof is complete. □

Theorem 3.2. There exists a constant $M > 0$ such that $S(t) \leqslant M, x(t) \leqslant M$ for each solution $(S(t), x(t))$ of (1.2) with all t large enough.

Proof. Suppose $(S(t), x(t))$ is any solution of (1.2). Let $V(t) = \delta S(t) + x(t)$. Then $V \in V_0$ and

$$\begin{cases} D^+V(t) = -\delta QS(t) - Qx(t) = -QV(t), & t \neq nT. \\ V(nT^+) = V(nT) + \delta\tau S^0, & n = 1, 2, \ldots \end{cases}$$

Obviously, we can choose $K > 0$ such that

$$\begin{aligned} &D^+V(t) \leqslant -QV(t) + K, && t \neq nT, \\ &V(nT^+) = V(nT) + \delta\tau S^0, && n = 1, 2, \ldots \end{aligned}$$

By Comparison Theory, we have

$$V(t) \leqslant \left(V(0^+) - \frac{K}{Q}\right)\mathrm{e}^{-Qt} + \frac{\delta\tau S^0(1-\mathrm{e}^{-nQT})}{1-\mathrm{e}^{-QT}}\mathrm{e}^{-Q(t-nT)} + \frac{K}{Q},$$
$$\times t \in (nT, (n+1)T].$$

Therefore, $V(t)$ is ultimately bounded by a constant and there exists a constant $M > 0$ such that $S(t) \leqslant M, x(t) \leqslant M$ for any solution $(S(t), x(t))$ of system (1.2) with all t large enough. The proof is complete. □

Now we investigate the permanence of the system (1.2).

Theorem 3.3. System (1.2) is permanent if

$$\int_0^T \frac{\mu_m S^*(t)}{K_m + S^*(t)}\mathrm{d}t > QT, \text{ i.e. } \frac{\mu_m}{Q}\ln\frac{K_m(1-\mathrm{e}^{-QT}) + \tau S^0}{K_m(1-\mathrm{e}^{-QT}) + \tau S^0\mathrm{e}^{-QT}} > QT.$$

Proof. Suppose $(S(t), x(t))$ is any solution of (1.2) with $(S(0^+), x(0^+)) > 0$. From Theorem 3.2, we can assume $S(t) \leqslant M, x(t) \leqslant M$ for $t \geqslant 0$. Choose $\epsilon_1 > 0$ small enough such that

$$m_1 = \frac{\tau S^0\mathrm{e}^{-QT}}{1-\mathrm{e}^{-QT}} - \epsilon_1 > 0 \text{ and } \sigma_0 = \frac{m_1\mu_m}{K_m + m_1} - Q < 0.$$

It follows from lemmas 2.2 and 2.3 that $S(t) > m_1$ for all t large enough.

Next, we prove that there exists an $m_2 > 0$ such that $x(t) > m_2$ for all t large enough in two steps.

Step 1. Since $\int_0^T \frac{\mu_m S^*(t)}{K_m+S^*(t)}\mathrm{d}t > QT$, we can choose $m_3 > 0, \epsilon_2 > 0$ small enough such that

$$\sigma = \int_0^T \left(\frac{\mu_m(\bar{y}(t) - \epsilon_2)}{K_m + (\bar{y}(t) - \epsilon_2)} - Q\right)\mathrm{d}t > 0, \text{ where}$$
$$\bar{y}(t) = \frac{\tau S^0 \exp\{-(Q + \frac{m_3\mu_m}{\delta K_m})(t-nT)\}}{1 - \exp\{-(Q + \frac{m_3\mu_m}{\delta K_m})T\}}, t \in (nT, (n+1)T].$$

We claim that $x(t) < m_3$ cannot hold for all $t \geqslant 0$, otherwise,

$$S'(t) \geqslant -S(t)\left(Q + \frac{m_3\mu_m}{\delta K_m}\right).$$

By lemmas 2.2 and 2.3, we have $S(t) \geqslant y(t)$ and $y(t) \to \bar{y}(t), t \to \infty$, where $y(t)$ is the solution of

$$\begin{aligned} y' &= -y(Q + \frac{m_3\mu_m}{\delta K_m}), && t \neq nT, \\ \triangle y &= y(t^+) - y(t) = \tau S^0, && t = nT, \\ y(0^+) &= S(0^+) > 0. \end{aligned} \tag{3.4}$$

Therefore, there exists a $T_1 > 0$ such that $S(t) \geqslant y(t) \geqslant \bar{y}(t) - \epsilon_2$ and

$$x' \geqslant x\left(-Q + \frac{\mu_m(\bar{y}(t) - \epsilon_2)}{K_m + (\bar{y}(t) - \epsilon_2)}\right) \tag{3.5}$$

for $t \geqslant T_1$.

Let $N_1 \in N$ and $N_1 T \geqslant T_1$. Integrating (3.5) on $(nT, (n+1)T]$, $n \geqslant N_1$, we have

$$x((n+1)T) \geqslant x(nT^+)\exp\left\{\int_{nT}^{(n+1)T}(-Q + \frac{\mu_m(\bar{y}(t) - \epsilon_2)}{K_m + (\bar{y}(t) - \epsilon_2)})\mathrm{d}t\right\} = x(nT)e^{\sigma}.$$

Then $x((N_1 + k)T) \geqslant x(N_1 T)e^{k\sigma} \to \infty$ as $k \to \infty$, which is a contradiction. Hence, there exists a $t_1 > 0$ such that $x(t_1) \geqslant m_3$.

Step 2. If $x(t) \geqslant m_3$ for all $t \geqslant t_1$, then the result is obtained. Hence, we need only to consider those solutions which leave the region $\Omega_1 = \{z \in R_+^2 : x(t) < m_3\}$ and enter it again. Let $t^* = \inf\{t \geqslant t_1 : x(t) < m_3\}$. Then $x(t) \geqslant m_3$ for $t \in [t_1, t^*)$ and $x(t^*) = m_3$ since $x(t)$ is continuous. Suppose $t^* \in [n_1 T, (n_1 + 1)T)$, $n_1 \in N$. Choose $n_2, n_3 \in N$ such that

$$n_2 T > T_2 = \frac{\delta K_m}{\delta Q K_m + m_3\mu_m}\ln\frac{M + \tau S^0}{\epsilon_2},\ e^{(n_2+1)\sigma_0 T}e^{n_3\sigma} > 1.$$

Let $T_3 = n_2 T + n_3 T$. We claim that there exists a $t_2 \in [(n_1+1)T, (n_1+1)T + T_3]$ such that $x(t_2) \geqslant m_3$. Otherwise, $x(t) < m_3$, $t \in [(n_1 + 1)T, (n_1 + 1)T + T_3]$. Consider (3.4) with $y((n_1 + 1)T^+) = S((n_1 + 1)T^+)$, we have

$$y(t) = \left(y((n_1 + 1)T^+) - \frac{\tau S^0}{1 - \exp\{-(Q + \frac{m_3\mu_m}{\delta K_m})T\}}\right)\mathrm{e}^{-(Q+\frac{m_3\mu_m}{\delta K_m})(t-(n_1+1)T)} + \bar{y}(t),$$

$t \in (nT, (n+1)T]$, $n_1 + 1 \leqslant n \leqslant n_1 + 1 + n_2 + n_3$.

Thus

$$|y(t) - \bar{y}(t)| < (M + \tau S^0)\exp\left\{-\left(Q + \frac{m_3\mu_m}{\delta K_m}\right)n_2 T\right\} < \epsilon_2$$

and

$$S(t) \geqslant y(t) \geqslant \bar{y}(t) - \epsilon_2,$$

or $t \in [(n_1 + 1 + n_2)T, (n_1 + 1)T + T_3]$, which implies (3.5) holds for $t \in [(n_1 + 1 + n_2)T, (n_1 + 1)T + T_3]$.

Integrating (3.5) on $[(n_1 + 1 + n_2)T, (n_1 + 1)T + T_3]$, we have

$$x((n_1 + 1 + n_2 + n_3)T) \geqslant x((n_1 + 1 + n_2)T)e^{n_3\sigma}.$$

It follows from the second equation of (1.2) that

$$x'(t) \geqslant x(t)\left[\frac{m_1\mu_m}{K_m+m_1} - Q\right] = \sigma_0 x(t).$$

Integrating it on $[t^*, (n_1+1+n_2)T]$, we have

$$x((n_1+1+n_2)T) \geqslant x(t^*)e^{\sigma_0(n_2+1)T} = m_3 e^{\sigma_0(n_2+1)T}.$$

Thus $x((n_1+1+n_2+n_3)T) \geqslant m_3 e^{n_3\sigma} e^{\sigma_0(n_2+1)T} > m_3$, which is a contradiction.

Let $\bar{t} = \inf\{t \geqslant t^* : x(t) \geqslant m_3\}$. Then $x(\bar{t}) \geqslant m_3$. For $t \in [t^*, \bar{t})$, we have $x(t) \geqslant x(t^*)e^{\sigma_0(t-t^*)} \geqslant m_3 e^{\sigma_0(1+n_2+n_3)T} \triangleq m_2$. For $t > \bar{t}$, the same arguments can be continued since $x(\bar{t}) \geqslant m_3$. Hence, we have $x(t) \geqslant m_2$ for all $t \geqslant t_1$. The proof is complete. □

In the following, we verify our main results by numerical simulation. Let $\mu_m = 2.0$, $K_m = 0.58$, $Q = 1, \delta = 10, T = 3$. Then whether the microorganism is extinct or not is determined completely by the input amount of the substrate for the fixed period T. Thus the critical value of the input amount of the substrate is $\tilde{S^0} \approx 2.47$. By theorem 3.1, If the input amount of the substrate $S^0 = 2.45 < 2.47$, the substrate $S(t)$ presents periodic oscillation as $t \to \infty$, the microorganism $x(t)$ is extinct as $t \to \infty$ (see figure 1). In this case, the substrate $S(t)$ and the microorganism $x(t)$ can not coexist (see the left figure of figure 3). When the input amount of the substrate $S^0 = 2.49 > 2.47$, by theorem 3.3, the microorganism $x(t)$ and the substrate $S(t)$ both present periodic oscillation and can coexist on a stable positive periodic solution as $t \to \infty$ (see figure 2 and the right figure of figure 3).

It is a significant and interesting problem whether the microbial culture is successful. We see that the microbial culture is successful if $\liminf_{t\to\infty} x(t) > x(0^+)$, the microbial culture is failed if $\limsup_{t\to\infty} x(t) < x(0^+)$. This fact is very close to practice. We can obtain the microorganism by increasing the input

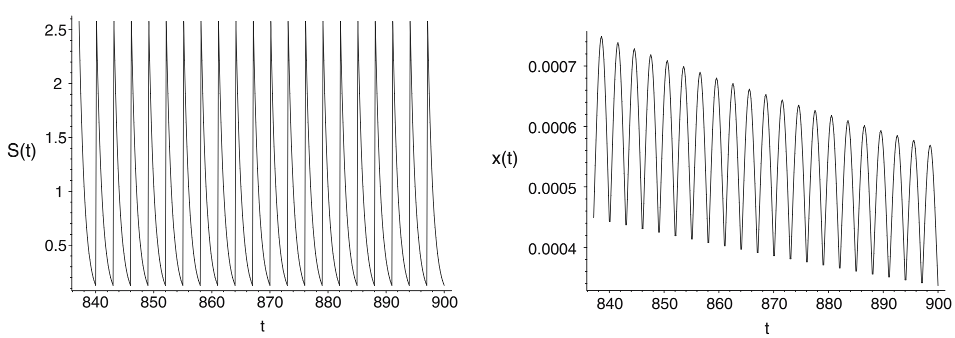

Figure 1. $\mu_m = 2.0$, $K_m = 0.58$, $Q = 1$, $\delta = 10$, $S^0 = 2.45$, $T = 3$. $(S(0^+), x(0^+)) = (1.5, 0.1)$. The left figure is time series of $S(t)$, the right one is time series of $x(t)$.

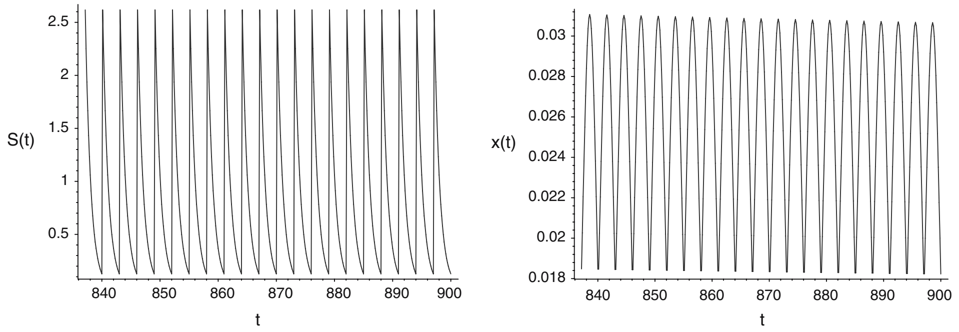

Figure 2. $\mu_m = 2.0$, $K_m = 0.58$, $Q = 1$, $\delta = 10$, $S^0 = 2.49$, $T = 3$. $(S(0^+), x(0^+)) = (1.5, 0.1)$. The left figure is time series of $S(t)$, the right one is time series of $x(t)$.

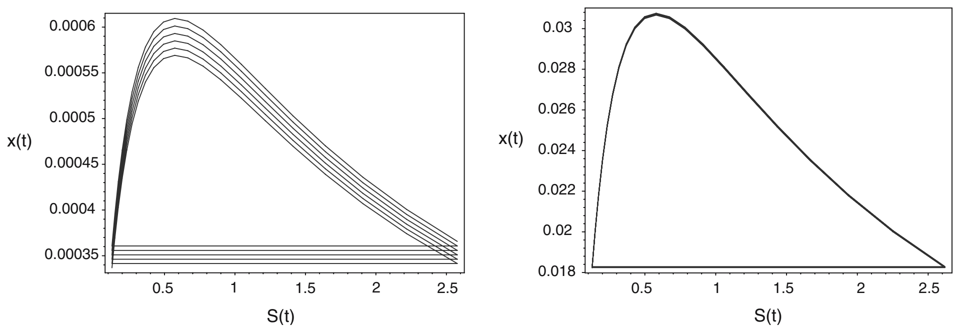

Figure 3. $\mu_m = 2.0$, $K_m = 0.58$, $Q = 1$, $\delta = 10$, $T = 3$. $(S(0^+), x(0^+)) = (1.5, 0.1)$. The left figure is phase portrait of $S(t)$ and $x(t)$ for $S^0 = 2.45$, the right one is phase portrait of $S(t)$ and $x(t)$ for $S^0 = 2.49$.

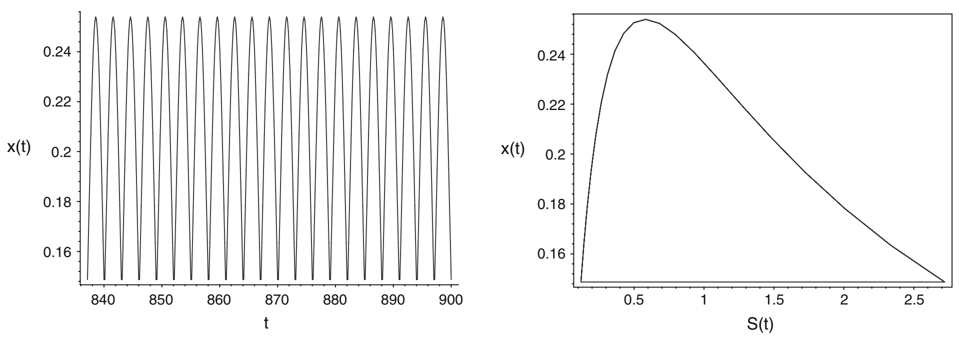

Figure 4. $\mu_m = 2.0$, $K_m = 0.58$, $Q = 1$, $\delta = 10$, $S^0 = 2.6$, $T = 3$. $(S(0^+), x(0^+)) = (1.5, 0.1)$. The left figure is time series of $x(t)$, the light one is phase portrait of $S(t)$ and $x(t)$.

amount of the substrate S^0. For example, if $S^0 = 2.6 > S^0_{\min} = 2.56$, then $\liminf_{t\to\infty} x(t) = 0.15 > x(0^+) = 0.1$ as $t \to \infty$ (see figure 4). Thus, the microorganism is obtained. Obviously, if both the continuous culture and the impulsive culture can obtain the microorganism, the latter is better than the former since the impulsive culture can save the substrate.

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